Sept 26, 2022
Week 4

2020A Adv. Cal. II

Let (x, y) be a point in $\mathbb{R}^2$. It can be uniquely expressed as

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned} \qquad (1)$$

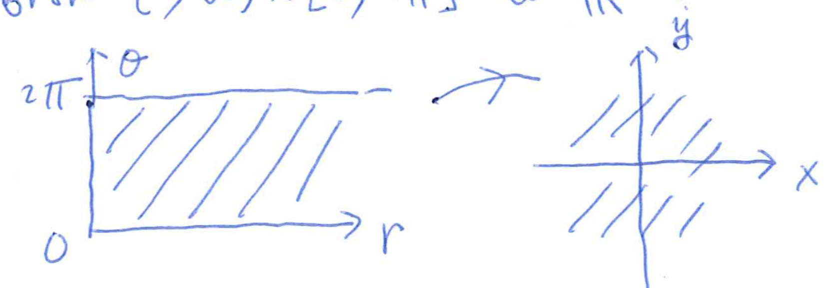
where $r > 0$ and $\theta \in [0, 2\pi)$ (or $[-\pi, \pi)$). The only exception is $(x, y) = (0, 0)$ where the representation

$$\begin{aligned} 0 &= 0\cos\theta \\ 0 &= 0\sin\theta \end{aligned} \quad \text{for all } \theta.$$

So the representation is not unique.

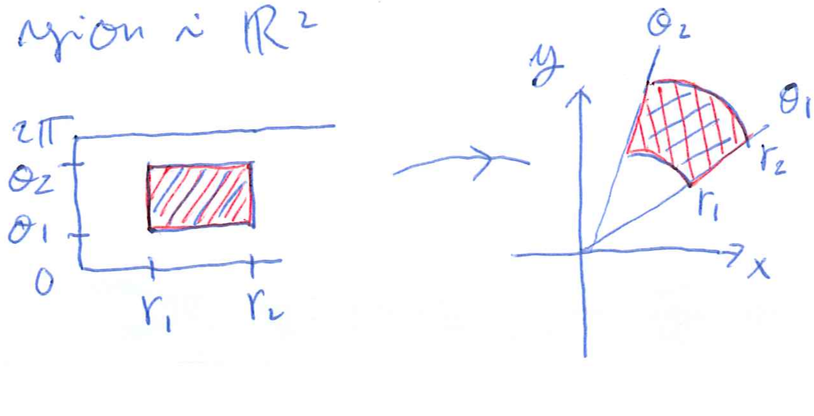
(r, θ) is called the polar coordinates of (x, y).

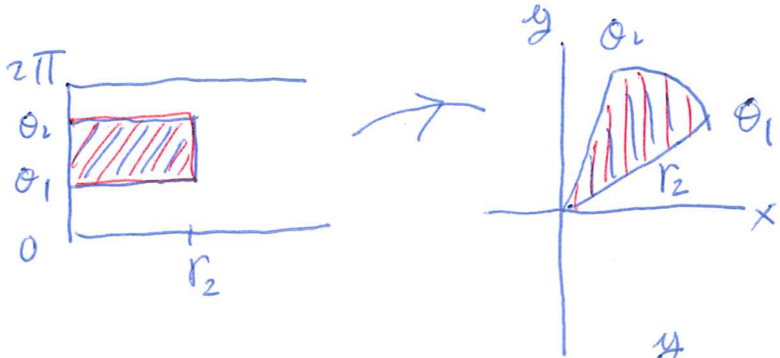
The map $(r, \theta) \mapsto (x, y)$ in (1) sets up a correspondence from $[0, \infty) \times [0, 2\pi]$ to $\mathbb{R}^2$.

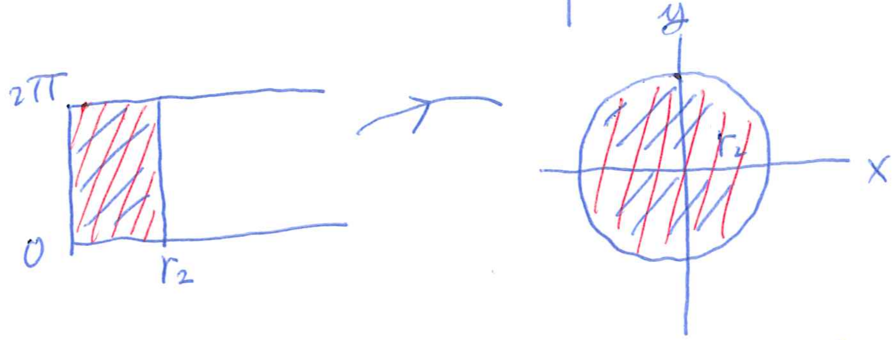


which is 1-1 onto $\mathbb{R}^2 \setminus (0,0)$ from $(0, \infty) \times [0, 2\pi)$ (or $(0, \infty) \times [-\pi, \pi]$)

In particular, it maps a rectangle in (r, θ) to a fan-shaped region in $\mathbb{R}^2$







Suppose now f is a piecewise continuous function on a fan-shaped region D given by $\theta_1 \le \theta \le \theta_2$, $r_1 \le r \le r_2$, we let

$$\hat{f}(r,\theta) \equiv f(r\cos\theta, r\sin\theta)$$

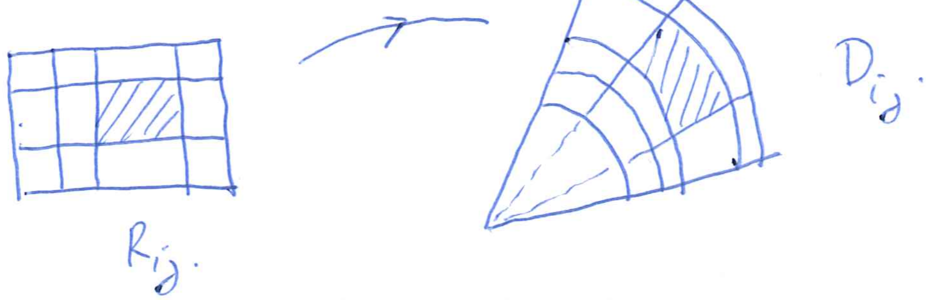
which is now a fcn on the rectangle $R = [r_1, r_2] \times [\theta_1, \theta_2]$.

Theorem

$$\iint_D f(x,y)\, dA(x,y) = \iint_R \hat{f}(r,\theta)\, r\, dA(r,\theta)$$

$$= \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

Pf: Let P be a partition of R whose subrectangles are R_{ij}.



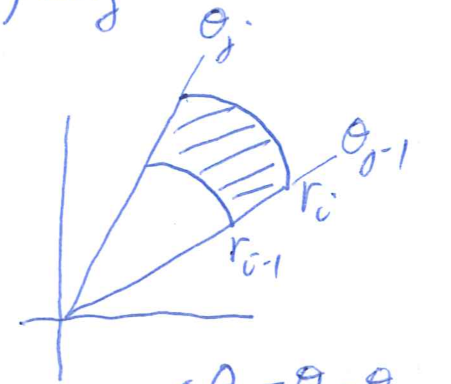
Under (1) we get a "generalized partition" on D

where sub-regions are D_{ij}. Let $(\bar{r}_i, \bar{\theta}_j)$ be the midpoint of R_{ij} and $(\bar{x}_i, \bar{y}_j)$ be the midpoint of D_{ij} related by (1). Consider the "generalized Riemann sum"

$$\sum f(\bar{x}_i, \bar{y}_j) |D_{ij}| , \qquad |D_{ij}| = \text{area of } D_{ij}$$

taking $(\bar{x}_i, \bar{y}_j)$ as tags. $|D_{ij}|$ is given by

$$\frac{1}{2} r_i^2 \Delta\theta_j - \frac{1}{2} r_{i-1}^2 \Delta\theta_j = \frac{1}{2}(r_i^2 - r_{i-1}^2)\Delta\theta_j$$
$$= \frac{1}{2}(r_i + r_{i-1})\Delta r_i \Delta\theta_j$$
$$= \bar{r}_i \Delta r_i \Delta\theta_j .$$



$\Delta\theta_j = \theta_j - \theta_{j-1}$
$\Delta r_i = r_i - r_{i-1}$

$$\therefore \sum_{i,j} f(\bar{x}_i, \bar{y}_j) |D_{ij}|$$
$$= \sum_{i,j} f(\bar{x}_i, \bar{y}_j) \bar{r}_i \Delta r_i \Delta\theta_j$$
$$= \sum_{i,j} \hat{f}(\bar{r}_i, \bar{\theta}_j) \bar{r}_i \Delta r_i \Delta\theta_j . \qquad (2)$$

On the other hand, letting $g(r,\theta) = \hat{f}(r,\theta) r$, we realize that (2) is a Riemann sum of $g(r,\theta)$. As $\|P\| \to 0$, (2)

$$\to \iint_R g(r,\theta)\, dA(r,\theta)$$
$$= \iint_R \hat{f}(r,\theta)\, r\, dA(r,\theta) .$$

However, (2) also

$$\to \iint_D f(x,y)\, dA(x,y), \text{ done. } \blacksquare$$

Will explain the last step in the next lecture.

Now, consider D described by

$$\rho_1(\theta) \leq r \leq \rho_2(\theta),$$
$$\theta_1 \leq \theta \leq \theta_2,$$

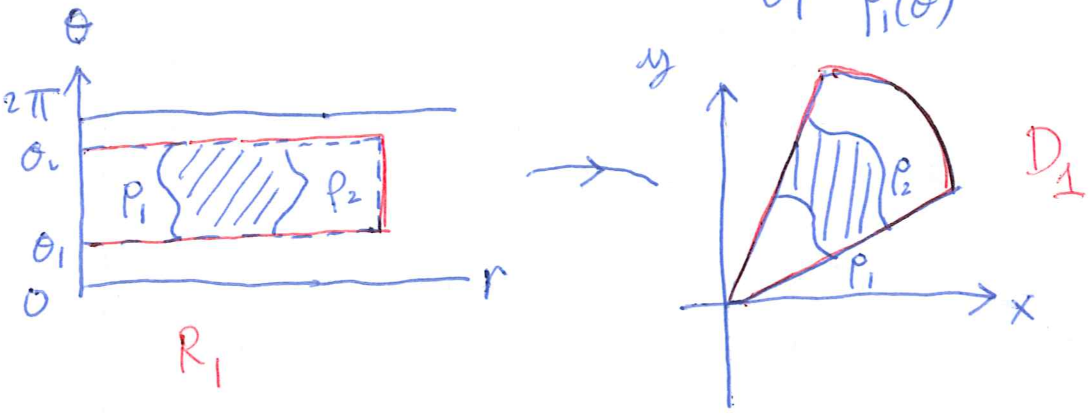
universal extension of f.

Then

$$\iint_D f(x,y)\,dA(x,y) = \iint_{D_1} \tilde{f}(x,y)\,dA(x,y)$$

$$= \iint_{R_1} \tilde{f}(r\cos\theta, r\sin\theta)\, r\, dA(r,\theta)$$

$$= \int_{\theta_1}^{\theta_2} \int_{\rho_1(\theta)}^{\rho_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$



the formula

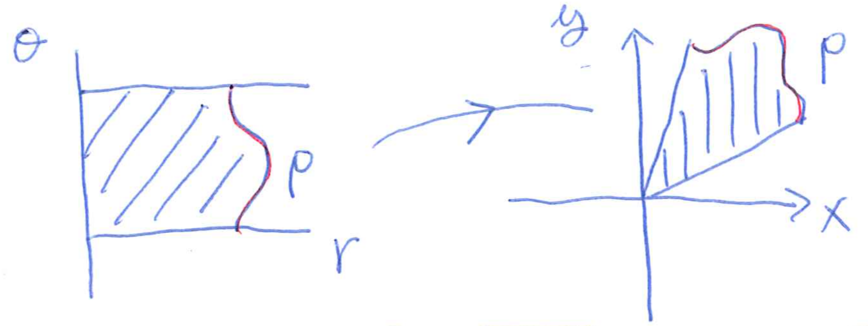
$$\iint_D f(x,y)\,dA(x,y) = \int_{\theta_1}^{\theta_2} \int_{\rho_1(\theta)}^{\rho_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

In the special case: D:

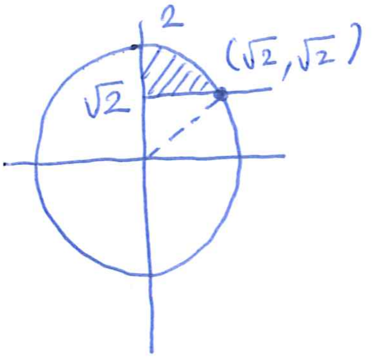
$$0 \leq r \leq \rho(\theta)$$
$$\theta_1 \leq r \leq \theta_2$$

the formula becomes

$$\iint_D f(x,y)\, dA(x,y) = \int_{\theta_1}^{\theta_2} \int_0^{\rho(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$



e.g. Let D be the region bounded by $x^2+y^2=4$, $y=\sqrt{2}$ and the y-axis. Describe it in polar coordinates.



ρ_1 is the horizontal line $y=\sqrt{2}$, ie

$r\sin\theta = \sqrt{2}$ or $r = \sqrt{2}/\sin\theta$

$\therefore \rho_1(\theta) = \sqrt{2}/\sin\theta$.

ρ_2 is the circle $x^2+y^2=4$, ie $r^2=4$, $r=2$

$\rho_2(\theta) = 2$ (a constant fcn)

$y=\sqrt{2}$ and $x^2+y^2=4$ intersect at $(\sqrt{2},\sqrt{2})$ so

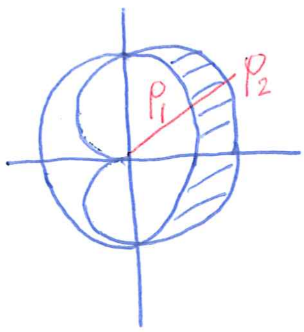
$\tan\theta_1 = \sqrt{2}/\sqrt{2} = 1$, ie $\theta_1 = \pi/4$.

clearly $\theta_2 = \pi/2$

$\therefore$ D : $\sqrt{2}/\sin\theta \le r \le 2$

$\pi/4 \le \theta \le \pi/2$.

e.g. Find the area of the region lying inside the cardioid $r = 1+\cos\theta$ but outside the circle $r=1$.



The cardioid and circle intersect at $(0,1)$ and $(0,-1)$, ie, $(1, \pi/2)$ and $(1, -\pi/2)$ in polar coordinat

D: $1 \le r \le 1+\cos\theta$
$-\pi/2 \le \theta \le \pi/2$.

By symmetry, area $= \iint_D 1\, dA$

$$= \int_{-\pi/2}^{\pi/2} \int_1^{1+\cos\theta} 1\, r\, dr\, d\theta$$

$$= 2\int_0^{\pi/2} \int_1^{1+\cos\theta} r\, dr\, d\theta$$

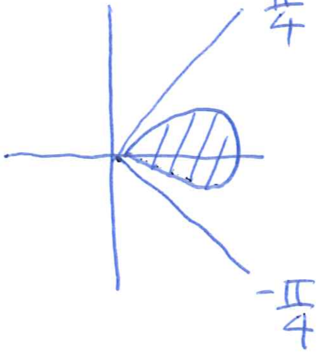
$$= 2\int_0^{\pi/2} \frac{1}{2} r^2 \Big|_1^{1+\cos\theta} d\theta$$

$\vdots$

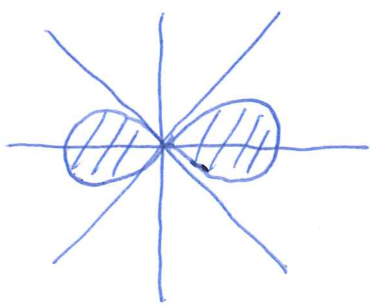
$$= 2 + \pi/4.$$

e.g. Find the area enclosed by the lemniscate $r^2 = 4\cos 2\theta$.

$\cos 2\theta$ is π-periodic, so it suffies to draw the graph on $[-\pi/2, \pi/2]$. When $2\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, ie, $\theta \in [-\frac{\pi}{4}, \frac{\pi}{4}]$, $\cos 2\theta \geq 0$ on $[-\pi/2, \pi/2] \setminus [-\frac{\pi}{4}, \frac{\pi}{4}]$, $\cos 2\theta < 0$ there is no graph. So



By π-periodicy, rotate it by π to get



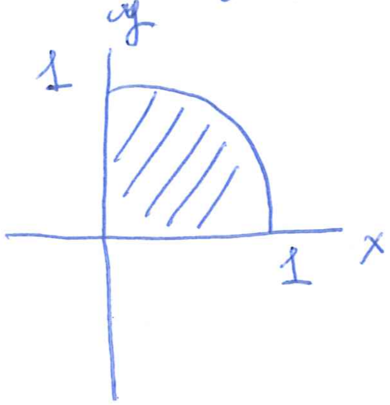
$$\text{area} = 2\int_{-\pi/4}^{\pi/4}\int_0^{\sqrt{4\cos 2\theta}} 1\, r\,dr\,d\theta$$

$$\vdots$$

$$= 4 \ \#$$

e.g. Evaluate the iterated integral $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx$ in polar coordinates.

D : $0 \le y \le \sqrt{1-x^2}$
$0 \le x \le 1$



in polar $0 \le r \le 1$
$0 \le \theta \le \pi/2$

$\therefore$ the integral

$$= \int_0^{\pi/2}\int_0^1 r^2\, r\,dr\,d\theta$$

$$\vdots$$

$$= \pi/8 \ \#$$

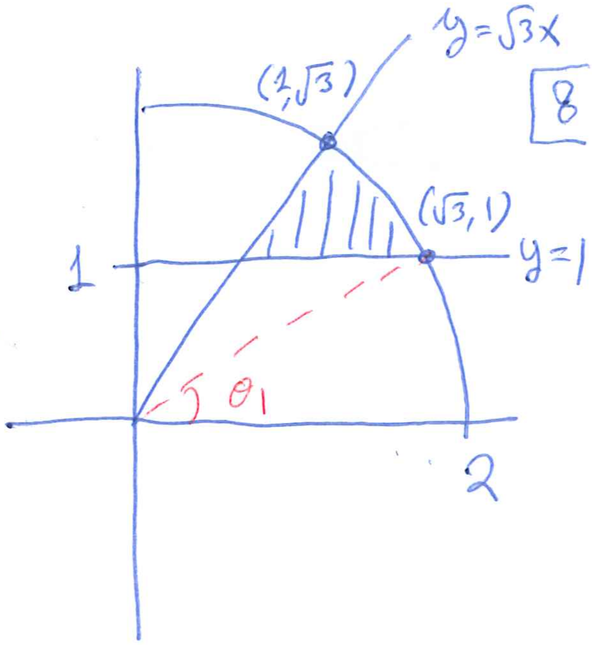
e.g. Find the volume of the solid bounded above by $z = 9-x^2-y^2$ and below by the unit disk in xy-plane.

$$\text{vol.} = \int_0^{2\pi}\int_0^1 (9-r^2)\,r\,dr\,d\theta$$

$$\vdots$$

$$= 17\pi/2$$

e.g. Let D be bounded by $x^2+y^2=4$, $y=1$, $y=\sqrt{3}x$.
Find its area.



θ_1 satisfies $\tan\theta_1 = \frac{1}{\sqrt{3}}$

$\theta_1 = \pi/6$

θ_2 satisfies $\tan\theta_2 = \frac{\sqrt{3}}{1}$

$\theta_2 = \pi/3$

$\therefore D: \ 1/\sin\theta \le r \le 2$

$\pi/6 \le \theta \le \pi/3$

$$\text{area} = \int_{\pi/6}^{\pi/3}\int_{1/\sin\theta}^{2} r\,dr\,d\theta$$

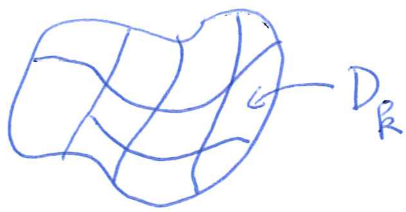
$$\vdots$$

$$= \frac{\pi-\sqrt{3}}{3} \ \#$$

(Cont'd)

Generalized partition on a region D

$D = \bigcup_{k=1}^{N} D_k$, D_k interior nonoverlap.



Generalized Riemann sum

$$S(f,P) = \sum_{k=1}^{N} f(p_k)\,|D_k|, \quad |D_k| \text{ area of } D_k.$$

$p_k \in D_k$ a tag

$$\|P\| = \max\{\operatorname{diam} D_k,\ k=1,\cdots,N\}$$

Theorem Let f be continuous in D. Then as $\|P\| \to 0$, the generalized Riemann sum

$$S(f, P) \longrightarrow \iint_D f \, dA .$$

Lemma 1 Let f be continuous in D. Let

$$m = \min_D f, \quad M = \max_D f.$$

For any a, $m \leq a \leq M$, there exists $p \in D$ s.t.

$$f(p) = a .$$

Pf. Let p_1 and p_2 be $f(p_1) = m$, $f(p_2) = M$. Connect p_1 to p_2 by a continuous path C. As the points running from p_1 to p_2, the values of f changes continuously. Since $a \in [m, M]$, then must a point p on C s.t. $f(p) = a$.

Lemma 2 Let f be continuous in D. Then exists $p \in D$ s.t.

$$f(p) = \frac{1}{|D|} \iint_D f .$$

Pf: $m \leq f(x,y) \leq M \ \forall (x,y) \in D$. Integrating over D:

$$\iint_D m \, dA \leq \iint_D f \leq \iint_D M \, dA, \text{ ie}$$

$$m|D| \leq \iint_D f \leq M|D|, \text{ ie}$$

$$m \leq \frac{1}{|D|} \iint_D f \leq M.$$

By Lemma 1, take $a = \frac{1}{|D|}\iint_D f$ to obtain the desired result. $\boxtimes$

$\boxtimes$ Pf of thm. Apply Lemma 2 to D_k,

$$\iint_D f = \sum_{k=1}^{N} \iint_{D_k} f$$

$$= \sum_{k=1}^{N} \frac{1}{|D_k|} \iint_{D_k} f \, |D_k|$$

$$= \sum_{k=1}^{N} f(p_k)\,|D_k| .$$

For any genralized Riemann sum, $\sum_{k=1}^{N} f(q_k)|D_k|$

$$\sum_{k=1}^{N} f(q_k)\,|D_k|$$

$$= \sum_{k=1}^{N} f(p_k)\,|D_k| + \sum_{k=1}^{N} \big(f(q_k) - f(p_k)\big)|D_k|$$

$$= \iint_D f + \sum_{k=1}^{N} \big(f(q_k) - f(p_k)\big)\,|D_k| .$$

As f is continuous on D, for any small $\varepsilon > 0$, we can find δ s.t. whenever $\|P\| < \delta$, (ie diam $D_k < \delta$, all k)

$$|f(q_k) - f(p_k)| < \varepsilon .$$

$$\therefore \quad \left| \sum_{k=1}^{N} f(q_k)\,|D_k| - \iint_D f \right| = \left| \sum_{k=1}^{N} \big(f(q_k) - f(p_k)\big)\,|D_k| \right|$$

$$\leq \sum_{k=1}^{N} |f(q_k) - f(p_k)|\,|D_k|$$

$$< \varepsilon \sum_{k=1}^{N} |D_k| = \varepsilon\,|D|$$

which means that

$$\sum_{k=1}^{N} f(q_k)\,|D_k| \longrightarrow \iint_D f \quad \text{as } \|P\| \to 0. \quad ▨$$